Continua in R*(General Topology and Geometric Topology)

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Continua in R^*

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Abstract. R^* is the Stone-Čech remainder of the real line. We prove that every decomposable continuum in R^* is a section of a standard continuum. Every indecomposable continuum in R^* is the union of a family of standard continua. About the general structures of continua in R^*, we have,

(1) Let C and D be continua in R^*. If one of them is indecomposable, then $C \subset D$, $D \subset C$ or $C \cap D = \phi$;

(2) R^* is hereditarily unicoherent, i.e., any intersection of continua in R^* is a continuum. Moreover, any intersection of indecomposable continua in R^* is indecomposable;

(3) The closure of the union of a chain of indecomposable continua in R^* is an indecomposable continuum;

(4) A point x of R^* is not a sub cutpoint iff $\{x\}$ is the intersection of a maximal chain of nondegenerate indecomposable continua in R^*;

(5) There is no Q-points in ω^* iff every composant of $\beta[0,\infty)-[0,\infty)$ is the union of a strictly increasing sequence of proper indecomposable subcontinua;

(6) The principle NCF is equivalent to the statement that $\beta[0,\infty)-[0,\infty)$ is the union of a strictly increasing sequence of proper indecomposable subcontinua.

Now we know that there are 9 different continua in R^*.

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Continua in R^*

By J. P. Zhu

Introduction.

We shall investigate continua in R^* in this paper. It is well-known that R^* is the topological sum of two indecomposable continua $\beta(-\infty,0]-(-\infty,0]$ and $\beta[0,\infty)-[0,\infty)$ [1]. Indeed, we can actually construct many continua in R^* by the following method. Let $\{I_n:n\in\omega\}$ be a discrete family of nondegenerate (faithfully indexed) closed intervals of R. For any nonprinciple ultrafilter u on ω, it is not difficult to show that the set

$$\bigcap\{cl_{\beta R}(\cup\{I_n:n\in A\}):A\in u\}$$

is a continuum (see, for example, [8]). These continua in R^* are called standard continua [8]. The first systematic study of standard continua was made by Mioduszewski in [10]. The most important fact discovered in [10] is that there is a natural partial order on every standard continuum. By this partial order, we can define layers and sections (See below). Layers are indecomposable continua ([13] and [15]) and sections are decomposable continua. Using these methods, Smith [12] proved that there are 8 different continua in R^* and the author [15] proved that infinitely many different indecomposable continua in R^* can be constructed by adding Cohen reals.

In this paper, we shall give a representation theorem for decomposable continua in R^*. Actually, we prove that every decomposable continua in R^* is a section of a standard continuum. As its corollaries, we have (1) Every indecomposable continuum in R^* is the union of a family of standard continua; (2) Let C and D be continua in R^*. If one of them is indecomposable, then $C\subset D$, $D\subset C$ or $C\cap D=\phi$; (3) R^* is hereditarily unicoherent, i.e., any intersection of continua in R^* is a continuum; (4) Any intersection of indecomposable continua in R^* is indecomposable; and (5) The closure of the union of a chain of indecomposable continua in R^* is an indecomposable continuum.

We can give a very simple explanation of the nonhomogeneity of R^*: Near points are sub cutpoints but never are larger points. This method was first used in [8] with a little bit more complicated notion. About sub cutpoints of R^*, we have the following characterization: A point x of R^* is a sub cutpoint iff $\{x\}$ is not the intersection of a maximal chain of non-degenerate indecomposable continua in R^*. The "only if" part was anounced by van Douwen in [4]. See also Corollary 5.4, which says that if we regard ω^* as a subspace of R^*, Q-points in ω^* are sub cutpoints in a very strong sense.

We have also noticed that there is a very closed relation between composants of $\beta[0,\infty)-[0,\infty)$ and Q-points in ω^*. We shall prove that there is no Q-points in ω^* iff every composant of $\beta[0,\infty)-[0,\infty)$ is the union of a strictly increasing sequence of proper indecomposable subcontinua. It is well-known that the statement that $\beta[0,\infty)-[0,\infty)$ is the unique composant of itself is equivalent to the principle NCF(Near Coherence of Filters, See [3]). Blass proved in [2] that NCF implies that there is no

Q-points in ω^*. Therefore, we have that NCF is equivalent to that $\beta[0,\infty)-[0,\infty)$ is the union of a strictly increasing sequence of proper indecomposable subcontinua. We would like to mention that every proper subcontinuun of $\beta[0,\infty)-[0,\infty)$ is nowhere dense in $\beta[0,\infty)-[0,\infty)$ since $\beta[0,\infty)-[0,\infty)$ is indecomposable (See, for example, [7]). At the end of this paper, we shall construct an indecomposable continuum in R^* which is not contained in [12]. Hence, there are at least 9 different continua in R^*.

§1. Preliminaries

A continuum is a compact connect Hausdorff space. A continuum X is nondegenerate if $|X|>1$ and a subcontinuum A of X is not proper if $A\in\{\phi,X\}$. A continuum is decomposable if it is the union of two proper subcontinua; Otherwise, it is indecomposable. We regard the empty set as an indecomposable continuum.

R is the real line. The Stone-Čech compactification of a space X is denoted by βX and the remainder $\beta X-X$ by X^*. We shall identify βX with the set of all ultrafilters of closed subsets of X, since we only consider the case of $X=R$ or a subspace of R in this paper. For each open set U of X, we let $O(U)=\{x\in\beta X:{}^{\exists}F\in x(F\subset U)\}$. Note that $\{O(U):U$ is open in $X\}$ is a base for βX.

If $e:\mathcal{I}\longrightarrow\mathcal{J}$ is a function from $\mathcal{I}$ to $\mathcal{J}$ and $\mathfrak{U}$ is an ultrafilter on $\mathcal{I}$, then the ultrafilter $\{A\subset\mathcal{J}:e^{-1}(A)\in\mathfrak{U}\}$ on $\mathcal{J}$ is denoted by $e(\mathfrak{U})$. If $(P,<)$ is a partial ordered set and $A,B\subset P$,

$A<B$ means that $a<b$ for any $(a,b)\in A\times B$.

We shall review some basic facts about standard continua in R^* in the rest of this section. Most of the results were proved in [10] and [12] in somewhat different forms.

Let Ω be the collection of all discrete infinite families of nondegenerate closed intervals of R. For $\mathcal{I}\in\Omega$ and a nonprinciple ultrafilter $\mathfrak{U}$ on $\mathcal{I}$, we let

$$M(\mathcal{I},\mathfrak{U})=\bigcap\{cl_{\beta R}(\cup\mathcal{F}):\mathcal{F}\in\mathfrak{U}\}.$$

It is not difficult to show that $M(\mathcal{I},\mathfrak{U})$ is a continuum. In fact, $M(\mathcal{I},\mathfrak{U})$ is a component of $(\cup\mathcal{I})^*$ (See, for example, [8])

Definition 1.1[8]. *A continuum* $B\subset R^*$ *is standard if there is* $\mathcal{I}\in\Omega$ *and a nonprinciple ultrafilter* $\mathfrak{U}$ *on* $\mathcal{I}$ *such that* $B=M(\mathcal{I},\mathfrak{U})$.

For any choice function f of $\mathcal{I}$, we let

$$f_{\mathfrak{U}}=\{F\subset R:F \text{ is closed and } \{I:f(I)\in F\}\in\mathfrak{U}\} \text{ and}$$

$$\mathfrak{C}(\mathcal{I},\mathfrak{U})=\{f_{\mathfrak{U}}:f \text{ is a choice function of } \mathcal{I}\}.$$

Recall that there is a natural partial order $<^{\mathcal{I}}_{\mathfrak{U}}$ on $M(\mathcal{I},\mathfrak{U})$ defined as follows: For any $x,y\in M(\mathcal{I},\mathfrak{U})$,

$$x<^{\mathcal{I}}_{\mathfrak{U}}y \text{ iff there are } F\in x \text{ and } H\in y \text{ such that } \{I\in\mathcal{I}:F\cap I<H\cap I\}\in\mathfrak{U}.$$

It is easy to see that $<_{\mathfrak{U}}^{\mathcal{J}}$ is a partial order. Moreover, if we restrict $<_{\mathfrak{U}}^{\mathcal{J}}$ on $\mathfrak{C}(\mathcal{J},\mathfrak{U})$, then it is a linear order ([10],see also Remark 2.6). For $x\in M(\mathcal{J},\mathfrak{U})$, we let

$$[x]_{\mathfrak{U}}^{\mathcal{J}}=\{y\in M(\mathcal{J},\mathfrak{U}):y \text{ is } <_{\mathfrak{U}}^{\mathcal{J}}\text{-incomparable with } x \text{ or } y=x\}.$$

$[x]_{\mathfrak{U}}^{\mathcal{J}}$ is called a layer of $M(\mathcal{J},\mathfrak{U})$(See [15]). As a good exercise, the reader is invited to check that $[x]_{\mathfrak{U}}^{\mathcal{J}}=\{x\}$ for any $x\in\mathfrak{C}(\mathcal{J},\mathfrak{U})$.

From now on, we shall omit the subscripts, if no confusion will occur.

Let $L(\mathcal{J},\mathfrak{U})=\{[x]:x\in M(\mathcal{J},\mathfrak{U})\}$. It is not difficult to show that $L(\mathcal{J},\mathfrak{U})$ is a partition of $M(\mathcal{J},\mathfrak{U})$ and the order on $L(\mathcal{J},\mathfrak{U})$ defined by $[x]<[y]$ iff $x<y$ is a linear order. For $x,y\in M(\mathcal{J},\mathfrak{U})$, let

$$(x,y)=\{z\in M(\mathcal{J},\mathfrak{U}):[x]<[z]<[y]\}$$

and

$$[x,y]=\{z\in M(\mathcal{J},\mathfrak{U}):[x]\leq[z]\leq[y]\}.$$

We endow $L(\mathcal{J},\mathfrak{U})$ with the order topology and define $\pi:M(\mathcal{J},\mathfrak{U})\longrightarrow L(\mathcal{J},\mathfrak{U})$ by $\pi(x)=[x]$. Since $(x,y)=\cup\{(a,b):a,b\in\mathfrak{C}(\mathcal{J},\mathfrak{U})$ and $x<a<b<y\}$ and (a,b) is obviously open in $M(\mathcal{J},\mathfrak{U})$ for any $a,b\in\mathfrak{C}(\mathcal{J},\mathfrak{U})$, we have

Theorem 1.2[10]. *The mapping* $\pi:M(\mathcal{J},\mathfrak{U})\longrightarrow L(\mathcal{J},\mathfrak{U})$ *defined as above is continuous.*

Definition 1.3. *The set* $[x,y]$ *is called a section of* $M(\mathcal{J},\mathfrak{U})$

for $x,y\in M(\mathcal{I},\mathfrak{U})$ *and* $x<y$. *Moreover, if* $x,y\in\mathfrak{C}(\mathcal{I},\mathfrak{U})$ *and* $x<y$, $[x,y]$ *is called a segment of* $M(\mathcal{I},\mathfrak{U})$.

If $[x,y]$ is a section of $M(\mathcal{I},\mathfrak{U})$, a layer of $M(\mathcal{I},\mathfrak{U})$ which is contained in $[x,y]$ is also called a layer of $[x,y]$ (See also Remark 2.6). The following result, which was proved by Smith [13] and the author [15], follows easily from Lemma 2.7 and Theorem 2.10.

Theorem 1.4. *Every layer is an indecomposable continuum.*

We collect some properties of sections.

Theorem 1.5. *Let* $[x,y]$ *be a section of the standard continuum* $M(\mathcal{I},\mathfrak{U})$, *we have:*

(1) $[x,y]$ *is a decomposable continuum irreducible from* x *to* y;

(2) *If* $[x',y']$ *is another section of* $M(\mathcal{I},\mathfrak{U})$, $[x,y]\cap[x',y']=[x^+,y^-]$, *where* $[x^+]=\max\{[x],[x']\}$ *and* $[y^-]=\min\{[y],[y']\}$. *If* $[x^+,y^-]\neq\phi$, *then* $[x,y]\cup[x',y']=[x^-,y^+]$, *where* $[x^-]=\min\{[x],[x']\}$ *and* $[y^+]=\max\{[y],[y']\}$;

(3) *Let* $C\subset[x,y]$ *be a nondegenerate subcontinuum, Then the following conditions are equivalent:*

(a) C *is not contained in any layer of* $M(\mathcal{I},\mathfrak{U})$;

(b) *There are* $x_0, y_0 \in [x,y]$ *such that* $x_0 < y_0$ *and* $C=[x_0, y_0]$;

(c) C *contains a cut point of* $[x,y]$;

(d) $C \cap \mathfrak{C}(\mathcal{F}, \mathfrak{U}) \neq \phi$;

(4) $[x,y]$ *admits an upper semicontinuous decomposition into indecomposable subcontinua so that the decomposition space is a Hausdorff arc*;

(5) *There are many nondegenerate layers in* $[x,y]$ *and every layer of* $[x,y]$ *is nowhere dense in* $[x,y]$;

(6) $[x,y] \cap \mathfrak{C}(\mathcal{F}, \mathfrak{U})$ *is dense in* $[x,y]$ *and* $\{(a,b) : a,b \in \mathfrak{C}(\mathcal{F}, \mathfrak{U})$ and $x<a<b<y\}$ *is a* π-base *for* $[x,y]$;

(7) *A point* $c \in [x,y]$ *is a cut point iff* $[c]=\{c\}$ *and* $x<c<y$. *In particular, a point* $c \in \mathfrak{C}(\mathcal{F}, \mathfrak{U})$ *is a cut point of* $[x,y]$ *iff* $x<c<y$;

(8) *Any point* $c \in \mathfrak{C}(\mathcal{F}, \mathfrak{U}) \cap [x,y]$ *is a* P-point *of* $[x,y]$;

(9) $[x,y]$ *is locally connected at each point in* $\mathfrak{C}(\mathcal{F}, \mathfrak{U}) \cap [x,y]$;

(10) $[x,y]$ *has the density of* 2^{ω}.

The proofs of (1) and (5)-(10) can be found in [12], where (10) follows from Lemma 2.1 in [12] by a standard tree argument

(See also [5],[10] and [15]). (2) is obvious since $(L(\mathcal{I},\mathfrak{U}),<)$ is a linearly ordered set. (3) can be proved by (6), (7) and Theorem 1.1. (4) follows from Theorem 1.1 and Theorem 1.4.

§2. Representations of continua in R^*

We start with the following observation.

Lemma 2.1. *If* K *is a closed subset of* R^* *and* W *is a neighbourhood of* K *in* βR, *there is an open set* U *of* R *such that* $K\subset O(U)\subset W$ *and* U *is the union of a discrete family of open intervals of* R.

Proof. Since K is compact, there are open sets V and V' of R such that $K\subset O(V')\subset cl_{\beta R}(V')\subset O(V)\subset W$. Let V be the union of a disjoint family $\{I_n : n\in\omega\}$ of open intervals. Then $\{I_n : I_n\cap V'\neq\phi$ and $n\in\omega\}$ is discrete. So $U=\bigcup\{I_n : I_n\cap V'\neq\phi$ and $n\in\omega\}$ is a desired one.

An immediate corollary of Lemma 2.1 is the following theorem due to van Mill and Mills [8].

Theorem 2.2[8]. *Every continuum in* R^* *is of one of the following forms:* $\beta(-\infty,0]-(-\infty,0]$, $\beta[0,\infty)-[0,\infty)$ *or the intersection of a family of standard continua.*

Proof. Let C be a continuum in R^*. Assume that C is neither $\beta(-\infty,0]-(-\infty,0]$ nor $\beta[0,\infty)-[0,\infty)$. By Lemma 2.1, $C=\bigcap\{O(U) : C\subset O(U)$ and $cl_R U=\cup\mathcal{I}$ for some $\mathcal{I}\in\Omega\}$. So $C=\bigcap\{(\cup\mathcal{I})^* : C\subset(\cup\mathcal{I})^*$ and

$\mathcal{I}\in\Omega\}$. This completes the proof since every component of $(\cup\mathcal{I})^*$ is a standard continuum.

Our goal in this section is to prove that every decomposable continuum in R^* is a section of a standard continuum. Recall that a decomposable continuum is the union of two proper subcontinua. We shall first consider the case of the union of two standard continua.

Definition 2.3. *Let* $M(\mathcal{I},\mathfrak{U})$ *and* $M(\mathcal{J},\mathcal{V})$ *be standard continua.* $M(\mathcal{I},\mathfrak{U})$ *and* $M(\mathcal{J},\mathcal{V})$ *are compatible if there are* $\mathcal{F}\in\mathfrak{U}$, $\mathcal{H}\in\mathcal{V}$ *and a bijection* $e:\mathcal{F}\longrightarrow\mathcal{H}$ *such that*

(1) $e(I)<e(J)$ *if* $I<J$;

(2) $I\cap e(J)\neq\phi$ *iff* $I=J$;

(3) $e(\mathfrak{U})=\mathcal{V}$, i.e., $\mathcal{V}=\{\mathcal{A}\subset\mathcal{J}:e^{-1}(\mathcal{A})\in\mathfrak{U}\}$.

If e *satisfies the additional condition*

(4) $I\subset e(I)$ *for any* $I\in\mathcal{I}$,

we say that $M(\mathcal{I},\mathfrak{U})$ *is identifiable in* $M(\mathcal{J},\mathcal{V})$.

Lemma 2.4. *Let* $M(\mathcal{I},\mathfrak{U})$ *and* $M(\mathcal{J},\mathcal{V})$ *be standard continua. Then,*

(a) *If* $M(\mathcal{I},\mathfrak{U})$ *is identifiable in* $M(\mathcal{J},\mathcal{V})$ *and* $M(\mathcal{J},\mathcal{V})$ *is identifiable in* $M(\mathcal{K},\mathcal{W})$, *then* $M(\mathcal{I},\mathfrak{U})$ *is identifiable in* $M(\mathcal{K},\mathcal{W})$;

(b) $M(\mathcal{I},\mathfrak{U})$ *is identifiable in* $M(\mathcal{J},\mathcal{V})$ *iff* $M(\mathcal{I},\mathfrak{U})$ *is a segment of* $M(\mathcal{J},\mathcal{V})$ *iff* $M(\mathcal{I},\mathfrak{U})$ *is a section of* $M(\mathcal{J},\mathcal{V})$ *iff* $M(\mathcal{I},\mathfrak{U})\subset M(\mathcal{J},\mathcal{V})$ *and* $M(\mathcal{I},\mathfrak{U})$ *is not contained in any layer of* $M(\mathcal{J},\mathcal{V})$ *iff*

$M(\mathcal{I},\mathfrak{U})\subset M(\mathcal{J},\mathcal{V})$ and $M(\mathcal{I},\mathfrak{U})\cap\mathfrak{C}(\mathcal{J},\mathcal{V})\neq\phi$.

Proof. (a) is obviously from the definition. For (b), by Theorem 1.5 (3), we need only to prove that if $M(\mathcal{I},\mathfrak{U})\subset M(\mathcal{J},\mathcal{V})$ and $M(\mathcal{I},\mathfrak{U})\cap\mathfrak{C}(\mathcal{J},\mathcal{V})\neq\phi$, then $M(\mathcal{I},\mathfrak{U})$ is identifiable in $M(\mathcal{J},\mathcal{V})$. Let f be a choice function of $\mathcal{J}$ such that $f_{\mathcal{V}}\in M(\mathcal{I},\mathfrak{U})$. Let $\mathcal{F}_1=\{I\in\mathcal{I}: I\subset\cup\mathcal{J}\}$, then $\mathcal{F}_1\in\mathfrak{U}$. Since $f_{\mathcal{V}}\in M(\mathcal{I},\mathfrak{U})$, $\mathcal{F}=\{I\in\mathcal{F}_1: {}^{\exists}J\in\mathcal{J}(f(J)\in I)\}\in\mathfrak{U}$. Since $\mathcal{J}$ is discrete, we have, for any $I\in\mathcal{F}$,

$$|\{J\in\mathcal{J}: J\cap I\neq\phi\}|=|\{J\in\mathcal{J}: I\subset J\}|=1.$$

It is obviously that $\mathcal{K}=\{J\in\mathcal{J}: {}^{\exists}I\in\mathcal{F}(I\subset J)\}$ belongs to $\mathcal{V}$ and the bijection $e:\mathcal{F}\longrightarrow\mathcal{K}$ defined by $e(I)=J$ iff $I\subset J$ satisfies the conditions (1)-(4) in Definition 2.3.

Lemma 2.5. *Let $M(\mathcal{I},\mathfrak{U})$ and $M(\mathcal{J},\mathcal{V})$ be standard continua. Then,*

(a) *$M(\mathcal{I},\mathfrak{U})$ and $M(\mathcal{J},\mathcal{V})$ are compatible iff $M(\mathcal{I},\mathfrak{U})\cap M(\mathcal{J},\mathcal{V})\neq\phi$ and there is a standard continuum $M(\mathcal{X},\mathcal{W})$ such that $M(\mathcal{I},\mathfrak{U})$ and $M(\mathcal{J},\mathcal{V})$ are both identifiable in $M(\mathcal{X},\mathcal{W})$;*

(b) *If $\mathfrak{C}(\mathcal{I},\mathfrak{U})\cap\mathfrak{C}(\mathcal{J},\mathcal{V})\neq\phi$, $M(\mathcal{I},\mathfrak{U})$ and $M(\mathcal{J},\mathcal{V})$ are compatible.*

Proof. (a) is trivial. Let us prove (b). Let f and g be choice functions of $\mathcal{I}$ and $\mathcal{J}$, respectively, such that $f_{\mathfrak{U}}=g_{\mathcal{V}}$. Let $A=f(\mathcal{I})\cap g(\mathcal{J})$. It is easily to see that $f^{-1}(A)\in\mathfrak{U}$ and $g^{-1}(A)\in\mathcal{V}$. Since $\mathcal{I}$ and $\mathcal{J}$ are discrete, we have, for any

$I_0 \in f^{-1}(A)$ and $J_0 \in g^{-1}(A)$,

$$|\{I \in f^{-1}(A) : I \cap J_0 \neq \phi\}| \leq 3 \quad \text{and} \quad |\{J \in g^{-1}(A) : J \cap I_0 \neq \phi\}| \leq 3.$$

We enumerate A as $\{x_n : n \in \omega\}$ so that $x_n < x_{n+1}$ for any $n \in \omega$. Let $A_i = \{x_{3n+i} : n \in \omega\}$ for each $i \leq 2$. Then $f^{-1}(A_i) \in \mathfrak{U}$ and $g^{-1}(A_i) \in \mathcal{V}$ for some $i \leq 2$. It is easily seen that the bijection $e : f^{-1}(A_i) \longrightarrow g^{-1}(A_i)$ defined by $e(I) = g^{-1}f(I)$ satisfies the conditions (1)-(3) in Definition 2.3.

Remark 2.6. *A standard continuum, of course, can be expressed by different* $(\mathcal{I}, \mathfrak{U})$ *and* $(\mathcal{J}, \mathcal{V})$. *However, we can see from Lemma 2.5 and Lemma 2.6 that the partial order, layers, sections,* $\mathfrak{C}(\mathcal{I}, \mathfrak{U})$, *compatiblity and identifiablity do not depend on the choice of* $(\mathcal{I}, \mathfrak{U})$.

Lemma 2.7. *Let* B_0 *and* B_1 *be standard continua with nonempty intersection. Then* $B_0 \cup B_1$ *is a standard continuum. Moreover, if* $B_0 - B_1 \neq \phi$, B_0 *is identifiable in* $B_0 \cup B_1$. *Therefore, if* $|B_0 \cap B_1| > 1$, $B_0 \cap B_1$ *is also a standard continuum.*

Proof. We assume that $B_0 - B_1$ and $B_1 - B_0$ are not empty. We prove that B_0 and B_1 are compatible. Our conclusions follow from Lemma 2.5 and Lemma 2.6. Let $B_0 = M(\mathcal{I}, \mathfrak{U})$ and $B_1 = M(\mathcal{J}, \mathcal{V})$. Let $\mathcal{F}_0 = \{I \in \mathcal{I} : I \not\subset \bigcup \mathcal{J}\}$ and $\mathcal{K}_0 = \{J \in \mathcal{J} : J \not\subset \bigcup \mathcal{I}\}$. Then $\mathcal{F}_0 \in \mathfrak{U}$ and $\mathcal{K}_0 \in \mathcal{V}$. It is easily seen that, for any $I_0 \in \mathcal{F}_0$ and $J_0 \in \mathcal{K}_0$, we have, $|\{I \in \mathcal{F}_0 : I \cap J_0 \neq \phi\}| \leq 2$ and $|\{J \in \mathcal{K}_0 : J \cap I_0 \neq \phi\}| \leq 2$. We enumerate $\mathcal{F}_0$ as

$\{I_n : n\in\omega\}$ so that $I_n < I_{n+1}$ for any $n\in\omega$ and $\mathcal{K}_0$ as $\{J_n : n\in\omega\}$ in the same way. Then $\{I_{2n+i} : n\in\omega\}\in\mathfrak{U}$ for some $i\leq 1$ and $\{J_{2n+j} : n\in\omega\}\in\mathcal{V}$ for some $j\leq 1$. Let $\mathcal{F}=\{I_{2n+i} : {}^{\exists}m\in\omega(J_{2m+j}\cap I_{2n+i})\neq\phi$ and $n\in\omega\}$ and $\mathcal{K}=\{J_{2n+j} : {}^{\exists}I\in\mathcal{F}(I\cap J_{2n+i})\neq\phi$ and $n\in\omega\}$. It is easily seen that $\mathcal{F}\in\mathfrak{U}$ and $\mathcal{K}\in\mathcal{V}$ and the bijection $e:\mathcal{F}\longrightarrow\mathcal{K}$ defined by $e(I)=J$ iff $I\cap J\neq\phi$ satisfies the conditions (1)-(3). This completes the proof.

By Theorem 2.2 , Lemma 2.7 and Theorem 1.5 (3), we have,

Corollary 2.8. *Let* C *be a continuum in* R^* *and* B *a standard continuum. If* $C\cap B$ *and* $B-C$ *are not empty, there is a standard continuum* $\tilde{B}$ *such that* $B\cup C\subset\tilde{B}$ *and* B *is identifiable in* $\tilde{B}$.

Although we can not prove in ZFC that for any $x,y\in\beta[0,\infty)-[0,\infty)$, there is a standard continuum B containing both x and y (See,[3]), the following result was proved in [16].

Lemma 2.9[16]. *Let* C *be a continuum in* R^*. *For any* $x,y\in C$, *if* U *and* V *are disjoint closed neighbourhoods of* x *and* y, *then there is a standard continuum* B *such that* $B\subset C$, $B\cap U\neq\phi$ *and* $B\cap V\neq\phi$.

Theorem 2.10. *Every decomposable subcontinuum is a section of a standard continuum.*

Proof. Let D be a decomposable continuum in R^* and $D=C_0\cup C_1$, where C_0 and C_1 are proper subcontinua of D. Let $x\in D-C_1$ and $y\in D-C_1$. Then, in D, there are a closed neighbourhood U of x and a closed neighbourhood V of y such that $x \in U \subset D-C_1$ and $y \in V \subset D-C_0$. By Lemma 2.9, there is a standard continuum B such that $B \cap U \neq \phi$, $B \cap V \neq \phi$ and $B \subset D$. Hence, the sets $B \cap C_0$, $B \cap C_1$, $B-C_1$ and $B-C_0$ are all nonempty. By Corollary 2.8, there are standard continua B_0 and B_1 such that $B \cup C_0 \subset B_0$, $B \cup C_1 \subset B_1$ and B is identifiable in both B_0 and B_1. By Lemma 2.5, there is a standard continuum $\tilde{B}$ such that B_0 and B_1 are identifiable in $\tilde{B}$. Therefore, $B \subset D \subset \tilde{B}$ and B is identifiable in $\tilde{B}$. Our theorem follows from Theorem 1.5(3).

Note that by Theorem 2.10 every decomposable continuum in R^* has all the properties which we list in Theorem 1.5. We would like to mention that Theorem 1.4 follows easily from Lemma 2.7 and Theorem 2.10 (Layers are obviously continua since they are the intersections of decreasing sequences of segments).

Corollary 2.11. *Let* C *and* D *be continua of* R^*. *If one of them is indecomposable, then* $C \subset D$, $D \subset C$ *or* $C \cap D = \phi$.

Proof. Suppose that $C-D$, $D-C$ and $C \cap D$ are all nonempty. Then $C \cup D$ is a decomposable continuum. So, $C \cup D$ is a section of a standard continuum. Assume that C is indecomposable. Then, by Theorem 1.5(3), C is contained in a layer of $C \cup D$. But layers are nowhere dense in any section. So $C \cup D = D$, which is a contradiction.

By Theorem 2.2 and Corollary 2.11, we have,

Corollary 2.12. *Every indecomposable continuum in* R^* *is the union of a family of standard continua.*

§3. The structures of continua in R^*

The following concept is well-known in continua theory.

Definition 3.1. *A space* X *is hereditarily unicoherent provided that any intersection of a family of continua in* X *is connected.*

Proposition 3.2. R^* *is hereditarily unicoherent.*

Proof. We need only to prove that any intersection of two continua is connected. Let C and D be continua in R^* and $C\cap D\neq\phi$. Assume that $C-D$ and $D-C$ are not empty. Then, $C\cup D$ is a decomposable continuum. By Theorem 2.10, $C\cup D$ is a section of a standard continuum. By our assumptions on C and D, neither C nor D is contained in a layer of $C\cup D$ since every layer is nowhere dense in $C\cup D$. The conclusion follows from Theorem 1.5(3) and (2).

Theorem 3.3. *Any intersection of a family of indecomposable continua in* R^* *is an indecomposable continuum.*

Proof. Recall that we regard the empty set as an indecomposable continuum. Suppose that $\mathscr{C}$ is a family of indecomposable continua in R^* and $\bigcap\mathscr{C}$ is decomposable. By Theorem 2.10, there is a standard continuum B such that $\bigcap\mathscr{C}$ is a section of B. We take another standard continuum B' so that B is identifiable in B' and $B'-B'\neq\phi$. For any $C\in\mathscr{C}$, by Corollary 2.11, we have that $C\subset B'$ or $B'\subset C$. Since C is indecomposable, if $C\subset B'$, then C is contained in a layer T of B' by Theorem 1.5(3). But B is identifiable in B'. So T is a layer of B. This is impossible. Hence, $B'\subset C$ for any $C\in\mathscr{C}$. $B'\subset\bigcap\mathscr{C}\subset B$. This is a contradiction.

We have proved that every decomposable continuum in R^* is a section of a standard continuum. It is natural to ask whether or not every indecomposable continuum is a layer of a standard continuum. However, the answer is no.

Corollary 3.4. *If $\mathscr{C}=\{C_\alpha:\alpha<\lambda\}$ is a strictly decreasing sequense of indecomposable continua in R^* and λ is a limit ordinal, $\bigcap\mathscr{C}$ is not a layer of any standard continuum.*

The following result will be used at the end of this paper to construct a continuum in R^* which is not contained in [12].

Corollary 3.5. *If $\mathscr{C}$ is a chain of indecomposable continua in R^*, $cl_{\beta R}(\bigcup\mathscr{C})$ is an indecomposable continuum.*

We conclude this section with two questions. We refer to [12] for more information.

Question 3.6. *Is every proper indecomposable subcontinuum of* $\beta[0,\infty)-[0,\infty)$ *homeomorphic to a layer of a standard continuum ?*

A positive answer to the following question gives a negative answer to Question 3.6 (See Theorem 5.9).

Question 3.7. *Does every layer have the property that every nonempty* G_δ*-set has nonempty interior ?*

§4. Sub cutpoints and nonhomogeneity of R^*

Definition 4.1[8]. *A point of a space is called a sub cutpoint if it is a cut point of some closed connected subspace.*

The following result follows from Theorem 2.10.

Proposition 4.2. *A point of* R^* *is a sub cutpoint iff it is a cut point of a standard continuum.*

Recall that a point $x\in R^*$ is near if $x\in cl_{\beta X}D$ for some closed discrete subset $D\subset R$. A point $x\in R^*$ is large if $x\notin cl_{\beta R}F$ for any closed set $F\subset R$ and $\mu(F)<\infty$, where μ is the Lebesgue measure. It is easily seen that near points are sub cutpoints. However, it follows from (Lemma1.3 and Proposition3.1) ~~Lemma 2.1~~ in [14] that if x is a cut point of a standard continuum, then for any $\varepsilon>0$, there is an $F\in x$ such that $\mu(F)<\varepsilon$. Therefore, we have,

Proposition 4.3. *Large points are not sub cutpoints.*

It is obvious that sub cutpoints are topologically invariant. So R^* is not homogeneous since near points are sub cutpoints but never are large points. This method was first appeared in [8]. But our presentation is simpler than the one in [8]. The "if" part of the following result was announced by van Douwen in [4].

Theorem 4.4. *A point* $x \in R^*$ *is not a sub cutpoint iff* $\{x\}$ *is the intersection of a maximal chain of indecomposable nondegenerate continua in* R^*.

Proof. Let $\mathscr{C}$ be a maximal chain of indecomposable non-degenerate continua in R^* and $\bigcap \mathscr{C} = \{x\}$. Suppose that x is a sub cutpoint. Then x is a cut point of a standard continuum B by Proposition 4.2. For any $C \in \mathscr{C}$, $C \subset B$ or $B \subset C$ by corollary 2.11. It is obvious that there is a $C \in \mathscr{C}$ such that $x \in C \subset B$. By Theorem 1.5(3), C is a section of B, hence, decomposable. This is a contradiction.

Assume that x is not a sub cutpoint. By Theorem 2.2, $\{x\}$ is the intersection of a family $\{B_\alpha : \alpha < \lambda\}$ of standard continua. Since x is not a cut point of B_α for any α, it follows from Theorem 1.5(7) that there is a layer T_α of B_α such that $x \in T_\alpha$ and $|T_\alpha| > 1$. Therefore, $\{x\} = \bigcap \{T_\alpha : \alpha < \lambda\}$. Since layers are indecomposable, $\{T_\alpha : \alpha < \lambda\}$ is a $\subset$-chain by Corollary 2.11. Hence,

$\{x\}$ is the intersection of a maximal chain of indecomposable nondegenerate continua in R^* by Zorn's Lemma.

The following question is the restatement of Question 65 in [6]. We refer to [5],[15] and [16] for more information.

Question 4.5. *Is any sub cutpoint near ?*

A positive answer to Question 4.5 gives a positive answer to the following question.

Question 4.6[8]. *Are near points topologically invariant in* R^*?

§5. Q-points and composants of $\beta[0,\infty)-[0,\infty)$

Recall that a point $p\in\omega^*$ is a Q-point if every finite-to-one function from ω to ω is one to one on a set in p. If $\mathcal{I}\in\Omega$ and $\mathfrak{U}$ is a nonprinciple ultrafilter on $\mathcal{I}$, we say that $\mathfrak{U}$ is a Q-ultrafilter if there is a bijection $i:\mathcal{I}\longrightarrow\omega$ such that $i(\mathfrak{U})$ is a Q-point in ω^*, equivalently, for any partition $\{\mathcal{I}_n : n\in\omega\}$ of $\mathcal{I}$ into finite subfamilies, there is $\mathcal{F}\in\mathfrak{U}$ such that $|\mathcal{F}\cap\mathcal{I}_n|\leq 1$ for any $n\in\omega$. Note that the existence of Q-points is independent with ZFC [9].

Proposition 5.1. *A point* $p\in\omega^*$ *is a* Q-point *if every finite-to-one monotone function from* ω *to* ω *is one-to-one on a set in* p.

Proof. Let $f:\omega\longrightarrow\omega$ be a finite-to-one function. We define a strictly increasing sequence $\{a_i : i\in\omega\}$ of integers as follows:

$$a_0=0;$$
$$a_1=\min\{n\in\omega : n>f(0)+\max(f^{-1}(0))\};$$
$$\cdots\cdots\cdots\cdots$$
$$a_i=\min\{n\in\omega : n>f(m)+\max(f^{-1}(m))+a_{i-1} \text{ for any } m\le a_{i-1}\}$$
$$\cdots\cdots\cdots\cdots$$

We define a function $h:\omega\longrightarrow\omega$ by $h(i)=n$ iff $a_{n-1}<i\le a_n$. Then h is finite-to-one and monotone. So there is a set $X\in p$ such that h is one-to-one on X. We enumerate X as $\{x_n : n\in\omega\}$ so that $x_n<x_{n+1}$ for any $n\in\omega$. Then f is one-to-one on $\{x_{3n+i}: n\in\omega\}$ for each $i\le 2$. Obviously, there is $i\le 2$ such that $\{x_{3n+i}: n\in\omega\}\in p$.

Theorem 5.2. *Let* $\mathcal{J}\in\Omega$ *and* $\mathfrak{U}$ *be a nonprinciple ultrafilter on* $\mathcal{J}$. *Then* $\mathfrak{U}$ *is a* Q-*ultrafilter iff for any standard continuum* B, $M(\mathcal{J},\mathfrak{U})\subset B$ *implies that* $M(\mathcal{J},\mathfrak{U})$ *is identifiable in* B.

Proof. Assume that $\mathfrak{U}$ is a Q-ultrafilter. Let $B=M(\mathcal{J},\mathcal{V})$ be the standard continuum containing $M(\mathcal{J},\mathfrak{U})$. It is easily seen that there is $\mathcal{F}_0\in\mathfrak{U}$ such that $\cup\mathcal{F}_0\subset\cup\mathcal{J}$. For each $J\in\mathcal{J}$, let $\mathcal{J}_J=\{I\in\mathcal{J}: I\subset J\}$. Then $\{\mathcal{J}_J : J\in\mathcal{J}\}\cup\{\{I\}: \forall J\in\mathcal{J}(I\not\subset J)\}$ is a partition of $\mathcal{J}$ into finite sets. Since $\mathfrak{U}$ is a Q-ultrafilter, there is $\mathcal{F}\subset\mathcal{F}_0$ such that $\mathcal{F}\in\mathfrak{U}$ and $|\mathcal{F}\cap\mathcal{J}_J|\le 1$ for any $J\in\mathcal{J}$. We define the function $e:\mathcal{F}\longrightarrow\mathcal{J}$ by $e(I)=J$ iff $I\subset J$. It is easily seen that e

satisfies the conditions (1)-(3) in Definition 2.3.

Suppose that $\mathfrak{U}$ is not a Q-ultrafilter. Let $i:\mathcal{J}\longrightarrow\omega$ be the bijection such that $i(I)<i(J)$ iff $I<J$. By Proposition 5.1, there is a finite-to-one and monotone function $f:\omega\longrightarrow\omega$ such that $f\circ i$ is not one-to-one on any $\mathcal{F}\in\mathfrak{U}$. For each $n\in\omega$, Let $J_n=[a_n,b_n]$, where $a_n=\inf(\cup(e^{-1}(f^{-1}(n))))$ and $b_n=\sup(\cup(e^{-1}(f^{-1}(n))))$. Then $\mathcal{J}=\{J_n:n\in\omega\}\in\Omega$. We define $e:\mathcal{J}\longrightarrow\mathcal{J}$ by $e(I)=J_{f\circ i(I)}$. Let $\mathcal{V}=e(\mathfrak{U})$. Then $M(\mathcal{J},\mathfrak{U})\subset M(\mathcal{J},\mathcal{V})$ but $M(\mathcal{J},\mathfrak{U})$ is not identifiable in $M(\mathcal{J},\mathcal{V})$.

It is easily seen from Theorem 2.2 that there is no maximal proper subcontinuum in $\beta[0,\infty)-[0,\infty)$. However, if we restrict to indecomposable subcontinua, the situation is quite different. We say that an indecomposable subcontinuum C of $\beta[0,\infty)-[0,\infty)$ is maximal if C is not properly contained in any proper subcontinuum of $\beta[0,\infty)$.

Corollary 5.3. *There is a Q-point in ω^* iff there is a maximal proper indecomposable subcontinuum in $\beta[0,\infty)-[0,\infty)$.*

Proof. Assume that C is a maximal proper indecomposable subcontinuum of $\beta[0,\infty)-[0,\infty)$. By Theorem 2.2 and Theorem 1.5(3), there is a standard continuum B such that C is contained in a layer of B. Let $B=M(\mathcal{J},\mathfrak{U})$. We claim that $\mathfrak{U}$ is a Q-ultrafilter, of course, which gives a Q-point in ω^*. Suppose that $\mathfrak{U}$ is not a Q-ultrafilter. By Theorem 5.2, there is a standard continuum B' such that $B\subset B'$ and B is not identifiable in B'.

By Lemma 2.4(b), B is contained in a layer of B'. This is a contradiction.

On the other hand, if $\mathfrak{U}$ is a Q-ultrafilter on $\mathcal{I}$ for some $\mathcal{I}\in\Omega$ and $\cup\mathcal{I}\subset R$, Then every layer of $M(\mathcal{I},\mathfrak{U})$ is a maximal proper indecomposable subcontinuum of $\beta[0,\infty)-[0,\infty)$ by Theorem 5.2.

If we regard ω^* as a subspace of R^*, then for any $p\in\omega^*$, the set $\{p\}$ is a maximal indecomposable subcontinuum of $\beta[0,\infty)-[0,\infty)$ iff p is a Q-point in ω^*. In other words, we have, see also Theorem 1.5(7),

Corollary 5.4. *Let* $p\in\omega^*$ *be a Q-point in* ω^* *and* C *a subcontinuum of* $\beta[0,\infty)-[0,\infty)$ *such that* $p\in C$. *If* C *is indecomposable, then* $C=\{p\}$ *or* $C=\beta[0,\infty)-[0,\infty)$; *If* C *is decomposable,* $\{p\}$ *is a layer of* C.

Recall that a subset C of a continuum K is a composant if, for some point $p\in C$, C is the set of all points x such that there is a proper subcontinuum of K containing both p and x. It is well-known that composants of an indecomposable continuum are disjoint (See, for example, [7]).

Proposition 5.5. *There is no Q-points in* ω^* *iff every composant of* $\beta[0,\infty)-[0,\infty)$ *is the union of a strictly increasing sequence of proper indecomposable subcontinua.*

Proof. Let C be a proper indecomposable subcontinuum of $\beta[0,\infty)-\beta[0,\infty)$. Then C is contained in a composant P of $\beta[0,\infty)-[0,\infty)$. Let $P=\bigcup\{C_\alpha:\alpha<\lambda\}$, where $\{C_\alpha:\alpha<\lambda\}$ is a strictly increasing sequence of proper indecomposable subcontinua of $\beta[0,\infty)-[0,\infty)$. Note that every composant is dense. So λ must be a limit ordinal. Hence, there is $\alpha<\lambda$ such that $C_\alpha-C$ and $C_\alpha\cap C$ are not empty. By Corollary 2.11, $C\subset C_\alpha$. Therefore, the "if" part follows from Corollary 5.3.

Assume that P is a composant of $\beta[0,\infty)-[0,\infty)$ and $\mathscr{C}=\{C_\alpha:\alpha<\lambda\}$ is a strictly increasing sequence of proper indecomposable subcontinua such that $C_\alpha\subset P$ and $P-\bigcup\mathscr{C}\neq\phi$. We prove that if there is no Q-points in ω^*, then there is a proper indecomposable subcontinuum $\tilde{C}$ such that $\bigcup\mathscr{C}\subset\tilde{C}$. Take a point $x\in P-\bigcup\mathscr{C}$ and a point $y\in C_{\alpha_0}$ for some $\alpha_0<\lambda$. Then there is a proper subcontinuum B of $\beta[0,\infty)-[0,\infty)$ containing both x and y. By Theorem 2.2, we can assume that B is standard. Since there is no Q-points, by Theorem 5.2, there is a standard continuum B' such that $B\subset B'$ and B is not identifiable in B'. It follows from Lemma 2.4 that there is a layer $\tilde{C}$ of B' such that $B\subset\tilde{C}$. So $\tilde{C}\cap C_\alpha\neq\phi$ for any $\alpha\geq\alpha_0$. By Corollary 2.11, $C_\alpha\subset\tilde{C}$ for any $\alpha<\lambda$ since $\tilde{C}-\bigcup\mathscr{C}\neq\phi$ and $\mathscr{C}$ is increasing. So $\bigcup\mathscr{C}\subset\tilde{C}$. The "only if " part of theorem follows from an induction.

Remark 5.6. *Suppose that* $M(\mathscr{I},\mathfrak{U})$ *is a standard continuum and* $\mathfrak{U}$ *is a Q-ultrafilter. Let* $D=\bigcup\{\mathfrak{C}(\mathscr{I},\mathscr{V}):M(\mathscr{I},\mathfrak{U})\subset M(\mathscr{I},\mathscr{V})\}$. *For any* $x,y\in D$, *we define*

$x<y$ *iff* $\exists M(\mathcal{G},\mathcal{V})(x,y\in M(\mathcal{G},\mathcal{V})$ *and* $x<^{\mathcal{G}}_{\mathcal{V}}y)$.

$x<y$ *is well-defined by Theorem 5.2 and Lemma 2.5. In fact,* $(D,<)$ *is a linearly ordered set.* D *is dense in* R^* *since* D *is dense in a composant of* R^* *by Lemma 2.7 and Theorem 1.5(6). Although the subspace topology and the order topology are coincident on each interval of* $(D,<)$, *they are different on* D. *About the existence of orderable dense subspace of* R^*, *we refer to* [14].

It is well-known that CH implies that $\beta[0,\infty)-[0,\infty)$ has 2^c many composants [11] and NCF is equivalent to that $\beta[0,\infty)-[0,\infty)$ is the unique composant of itself (See [3]). Blass [2] proved that NCF implies that there is no Q-points in ω^*. Therefore, we have

Corollary 5.7. NCF *is equivalent to that* $\beta[0,\infty)-[0,\infty)$ *is a union of a strictly increasing sequence of proper indecomposable subcontinua.*

We shall in conclusion construct an indecomposable continuum in R^*, which is not contained in [S1].

Lemma 5.8. *Let* $\pi:\omega\longrightarrow\omega$ *be the monotone function such that* $|\pi^{-1}(n)|=n$. *Then there is a sequence* $\{p_n:n\in\omega\}$ *of non Q-points such that* $\pi(p_n)=p_{n+1}$.

Proof. Let $\{X_n,\pi_n^{n+1}\}_{n\in\omega}$ be such that $X_n=\omega$ and $\pi_n^{n+1}:X_n\longrightarrow X_{n+1}$

is a copy of π for any $n\in\omega$. Let $\pi_n^m=\pi_{m-1}^m\circ\dots\circ\pi_{n+1}^{n+2}\circ\pi_n^{n+1}$ for $m>n$. For each $n\in\omega$, we let

$$\mathcal{F}_n=\{A\subset X_n:\exists i\in X_{n+1}\forall j>i(|(\pi_n^{n+1})^{-1}(j)-A|\le 1)\}.$$

It is easy to check that for any $n\in\omega$,

$$\mathcal{B}_n=\mathcal{F}_n\cup\{(\pi_n^m)^{-1}(A):\ A\in\mathcal{F}_m \text{ and } m>n\}$$

has finite intersection property. Moreover, $\mathcal{B}_m=\{\pi_n^m(A):A\in\mathcal{B}_n\}$ for any $m>n$. Let $p_0\in\omega^*$ be such that $\mathcal{B}_0\subset p_0$. Let $p_{n+1}=\pi_n^{n+1}(p_n)$ for $n\in\omega$. It is easily seen that $\pi:\omega\longrightarrow\omega$ witnesses that p_n is not a Q-point since $\mathcal{F}_n\subset p_n$. Therefore, $\{p_n:n\in\omega\}$ is the desired one.

Theorem 5.9. *There is an indecomposable subcontinuum* C *of* $\beta[0,\infty)-[0,\infty)$ *such that there is a nonempty* G_δ*-set of* C *which has empty interior and is the intersection of countably many open dense sets of* C.

Proof. Let $p_n\in\omega^*$ and $\pi:\omega\longrightarrow\omega$ as in Lemma 5.8. We take $\mathcal{I}_0\in\Omega$ and enumerate $\mathcal{I}_0$ as $\{I_n^0:n\in\omega\}$ so that $I_n^0<I_{n+1}^0$ for any $n\in\omega$. Let $I_n^0=[a_n^0,b_n^0]$ for any $n\in\omega$. We define $\iota_0:\omega\longrightarrow\mathcal{I}_0$ by $\iota_0(n)=I_n^0$ and $\mathfrak{U}_0=\iota_0(p_0)$. For each $n\in\omega$, let $k(n)=\min(\pi^{-1}(n))$ and $l(n)=\max(\pi^{-1}(n))$. Inductively, We define $\mathcal{I}_i\in\Omega$, $\iota_i:\omega\longrightarrow\mathcal{I}_i$ and $\mathfrak{U}_i$ from $\mathcal{I}_{i-1}$, ι_{i-1} and $\mathfrak{U}_{i-1}$ as follows:

$I_n^i=[a_n^i,b_n^i]$, where $a_n^i=a_{k(n)}^{i-1}$ and $b_n^i=b_{l(n)}^{i-1}$;

$\mathcal{I}_i=\{I_n^i:n\in\omega\}$;

$i_i:\omega\longrightarrow\mathcal{I}_i$ by $i_i(n)=I_n^i$; and

$\mathfrak{U}_i=i_i(p_i)$.

It is easily seen that $M(\mathcal{I}_0,\mathfrak{U}_0)\subset M(\mathcal{I}_1,\mathfrak{U}_1)\subset\ldots\subset M(\mathcal{I}_n,\mathfrak{U}_n)\subset\ldots$ and $M(\mathcal{I}_n,\mathfrak{U}_n)$ is not identifiable in $M(\mathcal{I}_{n+1},\mathfrak{U}_{n+1})$. By lemma 2.4, $M(\mathcal{I}_n,\mathfrak{U}_n)$ is contained in a layer T_n of $M(\mathcal{I}_{n+1},\mathfrak{U}_{n+1})$. Therefore, $\{T_n:n\in\omega\}$ is a strictly increasing sequence of indecomposable subcontinua in R^*. By Corollary 3.5, $C=cl_{\beta R}(\bigcup\{T_n:n\in\omega\})$ is an indecomposable continuum in R^*. It is well-known that every proper subcontinuum of an indecomposable continuum is nowhere dense [7]. So T_n is nowhere dense in C. Let $G=\bigcap\{C-T_n:n\in\omega\}$. Then G is a non-empty G_δ-set with empty interior since $\bigcup\{T_n:n\in\omega\}$ is dense in C.

Smith showed in [12] that $\beta[0,\infty)-[0,\infty)$ has 8 different subcontinua, among them six are decomposable. Other two are the degenerate continuum and the indecomposable continuum which is a Stone-Čech remainder of a locally compact, σ-compact and non-compact space, therefore, has the property that every nonempty G_δ-set has nonempty interior. By Theorem 5.7, we have

Corollary 5.10. *There are at least* 9 *different continua in* R^*.

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